Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Summer Examination-2018

Subject Name: Advanced Functional Analysis

Subject Code: 5SC04AFA1 **Branch: M.Sc. (Mathematics)**

Semester: 4 **Date: 24/04/2018** **Time: 10:30 To 01:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. Define: Hilbert Space. **(01)**
b. Give an example of inner product space which is not Hilbert space. **(01)**
c. Define: orthonormal set **(01)**
d. In standard notation prove that $\{e_n : n = 1,2,3, \dots\}$ is an orthonormal basis for l^2. **(02)**
e. State Bessel's inequality. **(02)**

Q-2 **Attempt all questions** **(14)**

a. Let X be an inner product space. For $x \in X$, define $\|x\| = \sqrt{< x, x >}$. Then prove that $\| \quad \|$ is a norm on X. Also prove that $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$. **(07)**

b. Let $K = C$ and $A: X \to X$ be a linear map. Then prove that **(07)**
$4 < A(x), y > = < A(x + y), x + y > - < A(x - y), x - y > +$
$i < A(x + iy), x + iy > -i < A(x - iy), x - iy >$

OR

Q-2 **Attempt all questions** **(14)**

a. State and prove Schwarz's inequality. When does equality hold? Justify your answer. **(07)**

b. State and prove Pythagoras theorem for inner product space. Let X be an inner product space and $\{x_1, x_{2,} \dots\}$ be orthogonal set in X and $k_1, k_2, \dots, k_n$ be scaler having absolute value 1. Then prove that **(07)**
$\|k_1x_1 + k_2x_2 + \cdots + k_nx_n\| = \|x_1 + x_2 + \cdots x_n\|$.

Q-3 **Attempt all questions** **(14)**

a. Let X be an inner product space and E be an orthonormal subset of X. Then prove the following **(07)**

(i) For each $x \in X$, $E_x = \{u \in E : < x, u > \neq 0\}$ is countable.
(ii) If $E_x = \{u_1, u_2, \dots\}$ then $< x, u_n > \to 0$as $n \to \infty$.

(iii) If X is a Hilbert space and $\sum_{n=1}^{\infty} < x, u_n > u_n$ converges to $y \in X$ then $(x - y) \perp E$.

b. State and prove Projection theorem. Also prove that completeness is essential in the projection theorem. **(07)**

OR

Q-3

a. Let H be a Hilbert space and $E \subset H$ be orthonormal set. Then prove that following are equivalent **(07)**

(i) E is orthonormal basis.

(ii) For each $x \in H, x = \sum_{n=1}^{\infty} < x, u_n > u_n$, where $E_x = \{u_1, u_2, \ldots.\}$

(iii) For each $x \in H$, $\|x\|^2 = \sum_{n=1}^{\infty} |< x, u_n >|^2$, where $E_x = \{u_1, u_2, \ldots.\}$

(iv) $L(E)$ is dense in H.

(v) For $x \in H$ and x is perpendicular to each element of E then $x = 0$.

b. State and prove Gram Schmidthorthonormalization process. **(07)**

SECTION – II

Q-4 **Attempt the Following questions** **(07)**

a. Define: Self adjoint operator **(01)**

b. If $A \in BL(H)$ is bounded below then $R(A^*) = H$. True or False. **(01)**

c. State: Riesz representation theorem **(01)**

d. Let H be a Hilbert space and $A, B \in BL(H)$ and $\lambda \in K$ then prove that $(A + B*=A*+B*$ **(02)**

e. If $A \in BL(H)$ then prove that if $\lambda \in W(A)$ if and only if $\bar{\lambda} \in W(A^*)$ **(02)**

Q-5 **Attempt all questions** **(14)**

a. Let H be a Hilbert space. Define $j_y: H' \to K$ for $y \in H$ by $j_y(f) = f(y), f \in H'$ then prove that j_y is continuous linear functional on H' and $\|j_y\| = \|y\|$. Also if $J: H \to H''$ by $J(y) = j_y$ then prove that J is an on to isometry isomorphism **(07)**

b. Let $A \in BL(H)$ be normal operator. Then prove the following **(07)**

(i) $k \in \sigma_e(A)$ iff $\bar{k} \in \sigma_e(A^*)$.

(ii) If $(A - kI)^2 x = 0$ then $k \in \sigma_e(A)$.

(iii) If k_1, k_2 are different eigenvalues of A and x_1, x_2 are corresponding eigenvectors then $x_1 \perp x_2$.

OR

Q-5

a. State and prove unique Hahn – Banach extension theorem. **(07)**

b. Define: Normal operator, Unitary operator and positive operator. Give an example of an operator which is normal but not unitary. **(07)**

Q-6 **Attempt all questions** **(14)**

a. Let $A \in BL(H)$. Then prove the following **(08)**

(i) A is unitary iff $\|A(x)\| = \|x\|$ and A is onto.

(ii) A is normal iff $\|A(x)\| = \|A^*(x)\| \; \forall \, x \in H$.

b. State Fuglede theorem. If A and B be normal operators such that $AB = BA$. Then prove that $A + B$ and AB are normal operators. Also prove that set of all normal operators are closed in $BL(H)$. **(06)**

OR

Q-6 **Attempt all Questions**

a. Define: Numerical range of bounded linear operator. **(07)**

Let H be a Hilbert space and $A \in BL(H)$ then prove that
(i) $\sigma_e(A) \subset W(A)$.
(ii) $\sigma(A) \subset \overline{W(A)}$.

b. Let $A \in BL(H)$ be a Hilbert Schmidt operator. Then prove the following **(07)**
(i) A is compact operator
(ii) A^* is a Hilbert Schmidt operator.